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OPTIMAL CONSUMPTION AND PORTFOLIO FOR AN INSIDER IN A MARKET WITH JUMPS

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Abstract. We consider the stochastic control problem in a financial market model driven by a Lévy process. In the market, we assume that there are two kinds of investors with different levels of information: a *uninformed agent* whose information coincides with the natural filtration of the price processes and an *insider* who has more information than the uninformed agent. Using forward integral techniques, we solve the optimal consumption and investment problem for the insider. We conclude by giving some examples.

Introduction

Consumption-portfolio problem in continuous time market models was first introduced by Merton (1969), (1971) with a strong assumption that stock prices were governed by Markovian dynamics with constant coefficients. This approach was based on the methods of stochastic dynamic programming. For a two assets market, he formulated the problem of choosing optimal portfolio selection and consumption rules as follows:

$$\max \ E\left\{ \int_0^T U(c(t),t)dt + g(X(T),T) \right\},$$

subject to the budget constraint, $c(t) \geq 0$, $X(0) = x$ and $X(t) > 0$ for all $t \in [0,T]$. Here $X(\cdot)$ represents the wealth process, $c(t)$ is the consumption per unit time at time t, U is assumed to be strictly concave *utility* function, g is the *bequest valuation* function which is usually assumed to be concave in terminal wealth $X(T)$. Recently many authors have used a martingale representation technique instead of dynamic programming methods: see Cox and Huang (1989), (1991), Karatzas, Lehoczky and Shreve (1987) and Pliska (1986). In incomplete markets, the theory was studied by He and Pearson (1991), Karatzas *et al.* (1991), Karatzas and Zitkovic (2003), Kramkov and Schachermayer (1999).

Using enlargement of filtration techniques, several authors are interested in financial markets in presence of informed agent. An early study of stochastic control problem for an insider who maximizes his expected logarithmic utility from terminal wealth and consumption in Brownian motion framework is the work of

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Karatzas and Pikovsky (1996). The first general study of insider trading based on forward integrals (without assuming enlargement of filtration) was done in Biagini and Øksendal (2006). See also Amendinger, Imkeller and Schweizer (1998). Subsequently, many authors used Malliavin calculus and forward integration to study the optimal portfolio of an insider. See e.g. Biagini and Øksendal (2005), Elliott, Geman and Korkie (1997), Grorud and Pontier (1998), Imkeller (2003), Kohatsu-Higa and Sulem (2006) and León, Navarro and Nualart (2003) for the Brownian motion case. An extension of forward integration to the case of compensated Poisson random measures was proposed by Di Nunno *et al.* (2005). This setting is used for the optimal portfolio problem by Di Nunno *et al.* (2006) and for the optimal consumption rate by Øksendal (2006).

In this paper, we extend the results of Di Nunno *et al.* (2006) and of Øksendal (2006) by considering both the optimal portfolio and consumption rate choices of an insider when his portfolio is allowed to anticipate the future. We formulate the associated optimal control problem as follows:

$$\max_{(c,\pi)} \mathbb{E}\left[\int_0^T e^{-\delta(t)} \ln c(t)dt + Ke^{-\delta(T)} \ln X^{(c,\pi)}(T)\right].$$

Here, $\delta(t) \geq 0$ is a given measurable process representing the discount rate, K is a nonnegative constant, $T > 0$ is a fixed terminal time and $X^{(c,\pi)}(\cdot)$ is the wealth process with control parameters (c, π). We solve this problem by adapting the ideas of Di Nunno *et al.* (2006) and Øksendal (2006) to our framework and obtain explicit solutions. Samuelson (1969) proved (for the discrete time case) and Merton (1969) confirmed that for logarithmic utility in a market without informed agent, the portfolio selection decision is independent of the consumption decision. Our paper is the generalization of this result in presence of insider. Indeed, if $\hat{\lambda}$ is an optimal relative consumption rate for the consumption problem without optimal portfolio, then it is also the optimal relative consumption rate for the optimal consumption and portfolio problem (see Theorem 4.4).

The most common approach to solve the insider's wealth optimization problem is assuming that the $\mathcal{F}_t$-measurable Brownian motion $B(t)$ is a semimartingale under the enlarged filtration. In this paper, instead of this assumption, we adopt the approach in Biagini and Øksendal (2006) and handle the problem by using forward integration. The principal result of this paper is that if there exist optimal portfolio and consumption, then $B(t)$ is a $\mathcal{G}_t$-semimartingale. Moreover, we show that it holds for Lévy processes (see Theorem 4.7).

This paper is organized as follows. In Section 2 we recall some mathematical preliminaries about forward integrals which are relevant to our calculations. In Section 3 the main problem is introduced and the explicit results are given in Section 4. In Section 5, we compare the optimal wealth process and the performance function of the informed and uninformed agents for some specific examples.

1. Preliminaries

In this section, we recall the forward integrals with respect to the Brownian motion and to the compensated Poisson random measure. For further information on the forward integration with respect to the Brownian motion, we refer to Russo and Vallois (1993), (2000) and (2007), Nualart (1986), Biagini and Øksendal (2005) and to Di Nunno *et al.* (2005) and (2006) for the forward integration with respect

to the compensated Poisson random measure.
Suppose (Ω, P) be a product of probability spaces such that

$$(\Omega, P) = (\Omega_B \times \Omega_\Lambda, P_B \otimes P_\Lambda)$$

on which are respectively defined a standard Brownian motion $\{B(t)\}_{0\leq t\leq T}$ and a compound Poisson process $\{\Lambda(t)\}_{0\leq t\leq T}$ such that

$$\Lambda(t) = \int_0^t \int_{\mathbb{R}} zN(dt, dz).$$

From now on, we define pure jump Lévy process, $\{\eta(t)\}_{0\leq t\leq T}$ such that

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z\tilde{N}(dt, dz)$$

where $\tilde{N}(dt, dz) = (N - \nu_{\mathcal{F}})(dt, dz) = N(dt, dz) - \nu_{\mathcal{F}}(dz)dt$ is a compensated Poisson random measure with finite Lévy measure $\nu_{\mathcal{F}}$.
We denote by $\{\mathcal{F}_t\}_{0\leq t\leq T}$ the augmented filtration generated by $(B(t), \Lambda(t))_{0\leq t\leq T}$. In particular $\{\mathcal{F}_t^B\}_{0\leq t\leq T}$ and $\{\mathcal{F}_t^\Lambda\}_{0\leq t\leq T}$ are the augmented filtrations generated by $B(\cdot)$ and $\Lambda(\cdot)$ respectively. Let $\{\mathcal{G}_t\}_{0\leq t\leq T}$ be the filtration such that

$$\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{F} \qquad \forall t \ \in [0, T]$$

where $T > 0$ is a fixed terminal time.
Let $\varphi(t, \omega)$ and $\psi(t, z, \omega)$ be $\mathcal{G}$-adapted processes. Then

$$\int_0^T \varphi(t, \omega)dB(t) \tag{1.0.1}$$

and

$$\int_0^T \int_{\mathbb{R}_0} \psi(t, z, \omega)\tilde{N}(dt, dz) \tag{1.0.2}$$

make no sense in the normal settings. We can handle this problem in two ways. First one is assuming that $B(t)$ and $\int_0^t \int_{\mathbb{R}_0} \psi(s, z, \omega)\tilde{N}(ds, dz)$ are semimartingales under the filtration $\mathcal{G}_t$, for $t \in [0, T]$. However, if we do not assume that $B(t)$ and $\int_0^t \int_{\mathbb{R}_0} \psi(s, z, \omega)\tilde{N}(ds, dz)$ are $\mathcal{G}_t$-semimartingales then it is natural to use forward integrals to make the integrals (1.0.1) and (1.0.2) well defined.

Note that by Doob-Meyer decomposition any $\mathcal{G}_t$-measurable semimartingale can be written as

$$X(t) = X(0) + M(t) + A(t)$$

where $M(t)$ is a local martingale and $A(t)$ is an adapted finite variation process. Then if we assume that $B(t)$ is a $\mathcal{G}_t$-semimartingale then we have :

$$B(t) = \hat{B}(t) + A_B(t), \qquad 0 \leq t \leq T$$

where $\hat{B}(t)$ is a $\mathcal{G}_t$-adapted Brownian motion and $A_B(t)$ is a $\mathcal{G}_t$-measurable finite variation continuous process. Equation (1.0.1) exists as a semimartingale integral,

$$\int_0^T \varphi(t, \omega)d\hat{B}(t) + \int_0^T \varphi(t, \omega)dA_B(t) = \int_0^T \varphi(t, \omega)dB(t). \tag{1.0.3}$$

Similarly, if we assume that $\eta(t)$ is a $\mathcal{G}_t$-semimartingale then we have :

$$\eta(t) = \hat{\eta}(t) + A_\eta(t), \qquad 0 \leq t \leq T$$

where $\hat{\eta}(t) := \int_0^t \int_{\mathbb{R}_0} z(N - \nu_{\mathcal{G}})(ds, dz)$ is a $\mathcal{G}_t$-martingale with Lévy measure $\nu_{\mathcal{G}}$ as the unique predictable compensator of $N(dt, dz)$ with respect to $\mathcal{G}_t$ and $A_\eta(\cdot)$ satisfies the same conditions as $A_B(\cdot)$. Equation (1.0.2) exists as a semimartingale integral,

$$\int_0^T \varphi(t,\omega) d\hat{\eta}(t) + \int_0^T \varphi(t,\omega) dA_\eta(t) = \int_0^T \varphi(t,\omega) d\eta(t).$$

Equivalently,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}_0} \psi(t,z,\omega)(N-\nu_{\mathcal{G}})(dt,dz) \quad &+ \quad \int_0^T \int_{\mathbb{R}_0} \psi(t,z,\omega)(\nu_{\mathcal{G}} - \nu_{\mathcal{F}})(dt,dz) \\ &= \quad \int_0^T \int_{\mathbb{R}_0} \psi(t,z,\omega)\tilde{N}(dt,dz). \end{aligned} \tag{1.0.4}$$

Note that since the integrals of the left hand side of equations (1.0.3) and (1.0.4) are well defined then the integrals of the right hand side are well defined too.

If we define a specific enlarged filtration such as $\mathcal{G}_t := \mathcal{F}_t^B \vee \sigma(B(T_0)), \;\; 0 \le t \le T$, $T_0 > T$ then by Jeulin (1980) (Theorem 3.23 p.46) $B(t)$ is automatically a semimartingale and $A_B(t)$ can be defined explicitly. Then

$$\hat{B}(t) := B(t) - \int_0^t \frac{B(T_0) - B(s)}{T_0 - s} ds, \quad 0 \le t \le T \tag{1.0.5}$$

where $\hat{B}(\cdot)$ is a $\mathcal{G}_t$-Brownian motion.
Similarly, if we define $\mathcal{G}_t := \mathcal{F}_t^\Lambda \vee \sigma(\eta(T_0)), 0 \le t \le T, T_0 > T$ then by Protter (2003) (Theorem 3, p.356) $\eta(t)$ is a $\mathcal{G}_t$-semimartingale and $A_\eta(t)$ can be defined explicitly. Then

$$\hat{\eta}(t) := \eta(t) - \int_0^t \frac{\eta(T_0) - \eta(s)}{T_0 - s} ds$$

is a $\mathcal{G}_t$-martingale for $0 \le t \le T$.

As we mentioned we can cope with this problem without assuming that the $\mathcal{F}_t$-Brownian motion $B(t)$ is a $\mathcal{G}_t$-semimartingale. In this case we need to use the forward integrals.

Definition 1.0.1. Let $\varphi(t,\omega)$ be a measurable process. The forward integral of φ with respect to Brownian motion is defined by

$$\int_0^\infty \varphi(t,\omega) d^- B(t) = \lim_{\varepsilon \to 0} \int_0^\infty \varphi(t,\omega) \frac{B(t+\varepsilon) - B(t)}{\varepsilon} dt$$

if the limit exists in probability. Then φ is called *forward integrable* with respect to Brownian motion. If the limit exists also in $L^2(P)$, we write $\varphi \in \mathbb{D}^B$.

In particular, we recall the following result.

Lemma 1.0.2. *Let φ be forward integrable and càglàd (i.e. left continuous with right limits). Then for any partition $0 = t_0 < t_1 < \ldots < t_N = T$*

$$\int_0^T \varphi(t,\omega) d^- B(t) = \lim_{|\triangle t| \to 0} \sum_j \varphi(t_j) \triangle B(t_j),$$

where $\triangle B(t_j) = B(t_{j+1}) - B(t_j)$ and $| \triangle t | = \sup_{j=0,\ldots,N-1} \triangle t_j$.

Remark 1.0.3. Let $\varphi \in \mathbb{D}^B$ be a càglàd process. If $B(t)$ is a semimartingale with respect to $\mathcal{G}_t$ then $\int_0^T \varphi(t,\omega)dB(t)$ exists as a semimartingale integral and

$$\int_0^T \varphi(t,\omega)d^-B(t) := \int_0^T \varphi(t,\omega)dB(t).$$

Let us now give the corresponding definition of forward integral with respect to the compensated Poisson random measure.

Definition 1.0.4. Let $\varphi(t,z) := \varphi(t,z,\omega)$, $t \in \mathbb{R}_+$, $z \in \mathbb{R}_0$ be a measurable random field. The forward integral of $\varphi(t,z)$ with respect to the compensated Poisson random measure is defined by

$$\int_0^\infty \int_{\mathbb{R}_0} \varphi(t,z)\tilde{N}(d^-t,dz) = \lim_{n\to 0} \int_0^\infty \int_{U_n} \varphi(t,z)\,\tilde{N}(dt,dz)$$

if the limit exists in probability. Here, U_n is an increasing sequence of compact sets where $U_n \subseteq \mathbb{R}_0$, $\nu_{\mathcal{F}}(U_n) < \infty$ and $\bigcup_{n=1}^\infty U_n = \mathbb{R}_0$. Then, φ is called *forward integrable* with respect to Poisson random measure. If the limit exists in $L^2(P)$, we write $\varphi \in \mathbb{D}^{\tilde{N}}$.

Remark 1.0.5. Let $\varphi \in \mathbb{D}^{\tilde{N}}$ be càglàd. If $\int_0^T \int_{\mathbb{R}_0} \varphi(t,z,\omega)\tilde{N}(dt,dz)$ is a semimartingale with respect to $\mathcal{G}_t$ then

$$\int_0^T \int_{\mathbb{R}_0} \varphi(t,z,\omega)\tilde{N}(d^-t,dz) := \int_0^T \int_{\mathbb{R}_0} \varphi(t,z,\omega)\tilde{N}(dt,dz).$$

The last result we establish in this section is the Itô formula for forward integrals. We first define what is a forward process.

Definition 1.0.6. A *forward process* is a measurable stochastic function $X(t)$, $t \in [0,T]$, that admits the representation

$$X(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \beta(s)d^-B(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(d^-s,dz)$$

where $\displaystyle\int_0^T |\alpha(s)| + \beta(s)^2 ds < \infty$, $\gamma(t,z)$ is continuous in z around zero for a.a. (t,ω) and such that

$$\int_0^t \int_{\mathbb{R}_0} |\gamma(s,z)|^2 \nu_{\mathcal{F}}(ds,dz) < \infty \text{ for a.a. } (t,\omega).$$

Moreover, $\beta(\cdot)$ and $\gamma(\cdot,\cdot)$ are forward integrable with respect to Brownian motion and compensated Poisson random measure. A shorthand notation for this is

$$d^-X(t) = \alpha(t) + \beta(t)d^-B(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(d^-t,dz). \tag{1.0.6}$$

Theorem 1.0.7. ***(Itô formula for forward integrals).***
Let $X(t)$ be a forward process of the form (1.0.6) and define $Y(t) = f(X(t))$ for

any $f \in \mathbb{C}^2(\mathbb{R})$. Then $Y(t)$ is also a forward process and

$$d^-Y(t) = \Big[f'(X(t))\alpha(t) + \frac{1}{2}f''(X(t))\beta(t)^2 + \int_{\mathbb{R}_0}\Big\{f(X(t^-)+\gamma(t,z))$$

$$- f(X(t^-)) - f'(X(t^-))\gamma(t,z)\Big\}\nu_{\mathcal{F}}(dz)\Big]dt + f'(X(t))\beta(t)d^-B(t)$$

$$+\int_{\mathbb{R}_0}\big(f(X(t^-)+\gamma(t,z)) - f(X(t^-))\big)\tilde{N}(d^-t,dz),$$

where $f'(x)$ and $f''(x)$ are the first and second derivative of f with respect to x.

Proof. We refer to Russo and Valois (2000) for the proof of Brownian motion case and to Di Nunno *et al.* (2005) for the processes driven by Poisson random measure. □

2. The main problem

Assume there is a riskless and a risky asset in an arbitrage-free financial market. The price per unit of the riskless asset is denoted by $S_0(\cdot)$ and satisfies the following O.D.E.

$$\begin{aligned} dS_0(t) &= r(t)S_0(t)dt,\\ S_0(0) &= 1. \end{aligned}$$

The risky asset has a price process $S_1(\cdot)$ defined by the following forward S.D.E.

$$\begin{aligned} dS_1(t) &= S_1(t^-)[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0}\gamma(t,z)\tilde{N}(dt,dz)],\\ S_1(0) &> 0. \end{aligned}$$

Assume that the coefficients $r(t) = r(t,\omega), \mu(t) = \mu(t,\omega), \sigma(t) = \sigma(t,\omega)$, $\gamma(t,z) = \gamma(t,z,\omega)$ satisfy the following conditions:

(1) $r(t), \mu(t), \sigma(t), \gamma(t,z)$ are $\mathcal{F}_t$ adapted càglàd processes.
(2) $\gamma(t,z) > -1 \quad dt \times \nu_{\mathcal{F}}(dz)$-a.e.
(3) $\int_0^T \{| r(t) | + | \mu(t) | + \sigma(t)^2 + \int_{\mathbb{R}_0}\gamma(t,z)^2\nu_{\mathcal{F}}(dz)\}dt < \infty$

In this paper we will consider an agent who wants to maximize his expected intertemporal utility of consumption and terminal value of wealth when the portfolio is adapted to the filtration $\mathcal{G}$. Remember that $\mathcal{G}_t$ is larger than the natural filtration $\mathcal{F}_t$ for all $t \in [0,T]$.

Let $\pi(t)$ be the fraction of the wealth invested in the stock (risky asset) at time t by an insider. Therefore, $\pi(t)$ is a $\mathcal{G}_t$-adapted process and it is natural to use the forward integration to make the integrals well defined. The corresponding wealth process $X^{(c,\pi)}$ of the insider is given by

$$\begin{aligned} d^-X^{(c,\pi)}(t) &= X^{(c,\pi)}(t^-)[\{r(t) + (\mu(t)-r(t))\pi(t)\}dt + \sigma(t)\pi(t)d^-B(t)\\ &\quad +\pi(t)\int_{\mathbb{R}_0}\gamma(t,z)\tilde{N}(d^-t,dz)] - c(t)dt \end{aligned}$$

with initial value $X^{(c,\pi)}(0) = x$. In this setting, we also assume that the insider has a consumption rate $c(t) = c_\lambda(t)$ defined by

$$c_\lambda(t) = \lambda(t)X^{(c_\lambda,\pi)}.$$

Then the corresponding wealth dynamic of the insider can be rewritten by the following forward S.D.E. for $t \in [0,T]$

$$\begin{aligned} d^- X^{(c_\lambda,\pi)}(t) &= X^{(c_\lambda,\pi)}(t^-)[\{r(t) - \lambda(t) + (\mu(t) - r(t))\pi(t)\}dt \\ &\quad +\sigma(t)\pi(t)d^- B(t) + \pi(t)\int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(d^- t, dz)], \end{aligned} \tag{2.0.7}$$

where the initial wealth is $X^{(c_\lambda,\pi)}(0) = x > 0$.
We will assume that the agent has a logarithmic utility. It is convenient to use such functions because it has iso-elastic marginal utility which means that an agent has the same relative risk-tolerance as toward the end of his life.

Problem 2.0.8. *Find the optimal consumption rate $c^*_\lambda(\cdot)$ and optimal portfolio $\pi^*(\cdot)$ for an insider subject to his budget constraint, i.e. the pair (c^*_λ, π^*) which maximizes the performance function given by*

$$J(c_{\lambda^*}, \pi^*) = \sup_{(c_\lambda,\pi)\in\mathcal{A}} \mathbb{E}\left[\int_0^T e^{-\delta(t)} \ln c_\lambda(t)dt + Ke^{-\delta(T)} \ln X^{(c_\lambda,\pi)}(T)\right],$$

where $\delta(t) \geq 0$ is a given $\mathcal{F}_\infty$-measurable process representing the discount rate, $T > 0$ is a fixed terminal time and $X^{(c_\lambda,\pi)}(\cdot)$ is the wealth process which satisfies the equation (2.0.7).

The set of admissible portfolios, $\mathcal{A}$, will be defined by the following definition which is quite similar to Di Nunno *et al.* (2005) and Øksendal (2006).

Definition 2.0.9. A $\mathcal{G}_t$ adapted stochastic process pair (c_λ, π) is called *admissible* if

(i) $\displaystyle\int_0^T \lambda(s)ds < \infty$ a.s.
(ii) $\pi(t)$ is càglàd.
(iii) $\pi(t)\sigma(t)$ and $\pi(t)\gamma(t,z)$ are forward integrable with respect to $B(t)$ and $\tilde{N}(dt,dz)$ respectively.
(iv) $1 + \pi(t)\gamma(t,z) > \varepsilon_\pi$ for a.a. (t,z) with respect to $dt \times \nu_{\mathcal{F}}(dz)$, for some $\varepsilon_\pi \in (0,1)$ depending on π.
(v) $\displaystyle\int_0^T \left\{\mid (\mu(s) - r(s))\pi(s) \mid +\sigma^2(s)\pi^2(s) + \int_{\mathbb{R}_0} \pi^2(s)\gamma^2(s,z)\nu_{\mathcal{F}}(dz)\right\} ds < \infty$ a.s.
(vi) $\displaystyle\mathbb{E}\left[\int_0^T e^{-\delta(t)}|\ln \lambda(t)|dt + Ke^{-\delta(T)}|\ln X^{c_\lambda}(T)|\right] < \infty.$
(vii) $\displaystyle\int_0^T e^{-\delta(u)}du + Ke^{-\delta(T)} \neq 0$ for all $t \in [0,T]$.

and we denote by $\mathcal{A}$ the set of all admissible pair (c_λ, π).

3. Characterization of the optimal consumption and investment choice

Let us first consider the following consumption problem. This was the problem considered in Øksendal (2006).

Problem 3.0.10. *Find $\lambda^* \in \mathcal{A}_\lambda$ such that*

$$J_1(c_{\lambda^*}) = \sup_{\lambda\in\mathcal{A}} J_1(c_\lambda)$$

where

$$J_1(c_\lambda) = E\left[\int_0^T e^{-\delta(t)} \ln c_\lambda(t)dt + K\, e^{-\delta(T)} \ln X^{(c_\lambda)}(T)\right].$$

In this problem, the cash flow $X^{(c_\lambda)}(t)$ is modelled by the following forward stochastic differential equation :

$$d^- X^{(c_\lambda)}(t) = X^{(c_\lambda)}(t^-)[\mu(t)dt + \sigma(t)d^- B(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(d^- t, dz)] - c_\lambda(t)dt,$$

with initial wealth $X^{(c_\lambda)}(0) = x > 0$.

Here $\mathcal{A}_\lambda$ is the set of admissible controls defined by Øksendal (2006). For completeness, we give the definition.

Definition 3.0.11. The set $\mathcal{A}_\lambda$ of admissible controls for Problem 3.0.10 is the set of $\mathcal{G}_t$-adapted processes $\lambda(t) \geq 0$ such that

$$\int_0^T \lambda(s)ds < \infty \text{ a.s.}$$

and

$$\mathbb{E}\left[\int_0^T e^{-\delta(t)}|\ln \lambda(t)|dt + Ke^{-\delta(T)}|\ln X^{(c_\lambda)}(T)|\right] < \infty.$$

The main result for this problem is the following one :

Proposition 3.0.12. *Define*

$$\hat{\lambda}(t) = \frac{\mathbb{E}[e^{-\delta(t)} \mid \mathcal{G}_t]}{\mathbb{E}[\int_t^T e^{-\delta(s)}ds + K\, e^{-\delta(T)} \mid \mathcal{G}_t]}; \quad t \geq 0. \tag{3.0.8}$$

If $\hat{\lambda} \in \mathcal{A}_\lambda$ then $\hat{\lambda} = \lambda^*$ is the optimal control for the Problem 3.0.10. If $\hat{\lambda} \notin \mathcal{A}_\lambda$ then an optimal control does not exist.

Proof. Proof can be found in Øksendal (2006). □

Since we use an iso-elastic utility function, this optimal consumption rate depends only on the discount rate. The other parameters in the economy such as interest rates or volatility do not appear. However, the consumption of the agent is depending on these coefficients through the wealth. The next step is to show that the optimal consumption rate found in Øksendal (2006) is also optimal for Problem 2.0.8.

Theorem 3.0.13. *Define $\hat{\lambda}$ as in the equation (3.0.8). Then $\hat{\lambda}$ is an* optimal relative consumption rate *independent of the portfolio chosen, in the sense that*

$$J(c_{\hat{\lambda}}, \pi) \geq J(c_\lambda, \pi)$$

for all c_λ and π such that $(c_{\hat{\lambda}}, \pi), (c_\lambda, \pi) \in \mathcal{A}$.

Proof. Choose $\lambda \in \mathcal{A}$. Then

$$J(c_\lambda, \pi) \quad = \quad \mathbb{E}\left[\int_0^T e^{-\delta(t)} \ln c_\lambda(t)dt + Ke^{-\delta(T)} \ln X^{(c_\lambda,\pi)}(T)\right]$$

Applying the Itô formula for the forward integrals for Lévy processes, the solution of equation (2.0.7) is as follows:

$$\begin{aligned} X^{(c_\lambda,\pi)}(t) &= x\exp\Bigg\{\int_0^t \{r(s)+(\mu(s)-r(s))\pi(s)-\lambda(s)-\frac{1}{2}\sigma(s)^2\pi(s)^2 \\ &\quad -\int_{\mathbb{R}_0}[\pi(s)\gamma(s,z)-\ln(1+\pi(s)\gamma(s,z))]\nu_{\mathcal{F}}(dz)\}ds \\ &\quad +\int_0^t \pi(s)\sigma(s)d^-B(s)+\int_0^t\int_{\mathbb{R}_0}\ln(1+\pi(s)\gamma(s,z))\tilde{N}(d^-s,dz)\Bigg\} \end{aligned}$$

So,

$$\begin{aligned} J(c_\lambda,\pi) &= \mathbb{E}\left[\int_0^T e^{-\delta(t)}\left(\ln\lambda(t)-\int_0^t\lambda(s)ds\right)-Ke^{-\delta(T)}\int_0^T\lambda(t)dt\right] \\ &\quad +L_\pi, \end{aligned}$$

where

$$L_\pi=\mathbb{E}\left[\int_0^T e^{-\delta(t)}h(t)+Ke^{-\delta(T)}h(T)\right]$$

and

$$\begin{aligned} h(t) &= \ln x+\int_0^t\Big\{(\mu(s)-r(s))\pi(s)+r(s)-\frac{1}{2}\pi(s)^2\sigma(s)^2 \\ &\quad -\int_{\mathbb{R}_0}[\pi(s)\gamma(s,z)-\ln(1+\pi(s)\gamma(s,z))]\nu_{\mathcal{F}}(dz)\Big\}ds \\ &\quad +\int_0^t\pi(s)\sigma(s)d^-B(s)+\int_0^t\int_{\mathbb{R}_0}\ln(1+\pi(s)\gamma(s,z))\tilde{N}(d^-s,dz). \end{aligned}$$

Since L_π does not depend on $\lambda(\cdot)$, we are in the same case as in Øksendal (2006), Theorem 3.3. Then $\hat{\lambda}(\cdot)$ is the optimal relative consumption rate for Problem 2.0.8 and is independent of the portfolio chosen.

□

Note that since the optimal relative consumption rate, $\lambda^*=\hat{\lambda}$ does not depend on optimal portfolio π, we can separate the main problem. Therefore, our problem turns to :

Problem 3.0.14. *Find π^* such that $(c_{\lambda^*},\pi^*)\in\mathcal{A}$ and*

$$\begin{aligned} J(c_{\lambda^*},\pi^*) &= \tilde{J}(\pi^*) \\ &= \sup_{(c_{\lambda^*},\pi)\in\mathcal{A}}\mathbb{E}\left[\int_0^T e^{-\delta(t)}\ln c_{\lambda^*}(t)dt+Ke^{-\delta(T)}\ln X^{(c_{\lambda^*},\pi)}(T)\right]. \end{aligned}$$

For all $(c_\lambda,\pi)\in\mathcal{A}$ let us define M_π and $Y_\pi(t)$ as follows:

$$\begin{aligned} M_\pi(t) &= \int_0^t\left\{\mu(s)-r(s)-\sigma(s)^2\pi(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)}\nu_{\mathcal{F}}(dz)\right\}ds \\ &\quad +\int_0^t\sigma(s)d^-B(s)+\int_0^t\int_{\mathbb{R}_0}\frac{\gamma(s,z)}{1+\pi(s,z)\gamma(s,z)}\tilde{N}(d^-s,dz). \end{aligned} \tag{3.0.9}$$

and

$$Y_\pi(t) = \int_0^t e^{-\delta(u)} M_\pi(u)du + M_\pi(t)\left(\int_t^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right) \tag{3.0.10}$$

Moreover, we make the following assumptions :

(A.1) $\forall (c_\lambda, \pi), (c_\lambda, \theta) \in \mathcal{A}$ with θ bounded there exists positive τ s.t. the family $\{| M_{\pi+\varepsilon\theta}(T) |\}_{0\le\varepsilon\le\tau}$ is uniformly integrable.

(A.2) For all $t \in [0,T]$ the process pair (c_λ, π) where $\pi(s) := \chi_{(t,t+h]}(s)\theta_0(\omega)$, with $h > 0$ and $\theta_0(\omega)$ a bounded $\mathcal{G}_t$-measurable random variable, belongs to $\mathcal{A}$.

Theorem 3.0.15. *Suppose (c_{λ^*}, π^*) is optimal for Problem 3.0.14. Then $Y_{\pi^*}(t)$ defined in* (3.0.10) *is a $\mathcal{G}_t$-martingale.*

Proof. Suppose that (c_λ^*, π^*) is optimal for the insider. We can choose $\beta(\cdot)$ such that $(c_\lambda^*, \beta) \in \mathcal{A}$. Then $(c_\lambda^*, \pi^*(\cdot) + y\beta(\cdot)) \in \mathcal{A}$, for all y small enough. Since $\tilde{J}(\pi^* + y\beta)$ is maximal at π^*, then we have

$$\frac{d}{dy}\tilde{J}(\pi^* + y\beta)|_{y=0} = 0,$$

which implies

$$\begin{aligned}
&\mathbb{E}\Bigg[\int_0^T e^{-\delta(s)}\Bigg(\int_0^s \beta(u)\{\mu(u) - r(u) - \pi^*(u)\sigma(u)^2 \\
&-\int_{\mathbb{R}_0}(\gamma(u,z) - \frac{\gamma(u,z)}{1+\pi^*(u)\theta(u,z)})\nu_{\mathcal{F}}(dz)\}du + \int_0^s \beta(u)\sigma(u)d^-B(u) \\
&+\int_0^s\int_{\mathbb{R}_0}\frac{\beta(u)\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)}\tilde{N}(d^-u,dz)\Bigg)ds \\
&+Ke^{-\delta(T)}\Bigg(\int_0^T \beta(u)\sigma(u)d^-B(u) + \int_0^T\int_{\mathbb{R}_0}\frac{\beta(u)\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)}\tilde{N}(d^-u,dz) \\
&+\int_0^T \beta(u)\{\mu(u) - r(u) - \pi^*(u)\sigma(u)^2 \\
&-\int_{\mathbb{R}_0}(\gamma(u,z) - \frac{\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)})\nu_{\mathcal{F}}(dz)\}du\Bigg)\Bigg] = 0.
\end{aligned} \tag{3.0.11}$$

Let us fix $t \in [0,T)$ and $h > 0$ such that $t+h \le T$. We can choose β of the form

$$\beta(s) = \chi_{(t,t+h]}(s)\beta_0 \tag{3.0.12}$$

where β_0 is a bounded $\mathcal{G}_t$-measurable random variable. Rewriting equation (3.0.11) using (3.0.12), we obtain :

$$\mathbb{E}\Bigg[\beta_0 \int_t^{t+h} e^{-\delta(s)} \left(\int_t^s \{\mu(u) - r(u) - \pi^*(u)\sigma(u)^2 - \int_{\mathbb{R}_0} \frac{\pi^*(u)\gamma(u,z)^2}{1+\pi^*(u)\gamma(u,z)} \nu_{\mathcal{F}}(dz)\} du \right.$$
$$\left. + \int_t^s \sigma(u) d^- B(u) + \int_t^s \int_{\mathbb{R}_0} \frac{\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)} \tilde{N}(d^-u, dz) \right) ds$$
$$+ \beta_0 \int_{t+h}^T e^{-\delta(s)} \left(\int_t^{t+h} \sigma(u) d^- B(u) + \int_t^{t+h} \int_{\mathbb{R}_0} \frac{\gamma(u,z)}{1+\pi^*(u)\gamma(u,z)} \tilde{N}(d^-u, dz) \right.$$
$$\left. + \int_t^{t+h} \left\{ \mu(u) - r(u) - \pi^*(u)\sigma(u)^2 - \int_{\mathbb{R}_0} \frac{\pi^*(u)\gamma(u,z)^2}{1+\pi^*(u)\gamma(u,z)} \nu_{\mathcal{F}}(dz) \right\} du \right) ds \Bigg]$$
$$+ \mathbb{E}\Bigg[K e^{-\delta(T)} \beta_0 \left(\int_t^{t+h} \left\{ \mu(u) - r(u) - \pi^*(u)\sigma(u)^2 - \int_{\mathbb{R}_0} \frac{\pi^*(u)\gamma(u,z)^2}{1+\pi^*(u)\gamma(u,z)} \nu_{\mathcal{F}}(dz) \right\} du \right.$$
$$\left. + \int_t^{t+h} \beta(u)\sigma(u) d^- B(u) + \int_t^{t+h} \int_{\mathbb{R}_0} \frac{\gamma(u,z)}{1+\gamma(u,z)\pi^*(u)} \tilde{N}(d^-u, dz) \right) \Bigg] = 0.$$

Let us define $M_\pi(\cdot)$ as in equation (3.0.9), then the above equation turns to :

$$\mathbb{E}\Bigg[\beta_0 \Bigg(\int_t^{t+h} e^{-\delta(s)} M_{\pi^*}(s) ds + M_{\pi^*}(t+h) \left(\int_{t+h}^T e^{-\delta(s)} ds + K e^{-\delta(T)} \right)$$
$$- M_{\pi^*}(t) \left(\int_t^T e^{-\delta(s)} ds + K e^{-\delta(T)} \right) \Bigg) \Bigg] = 0.$$

If we take now,

$$N_{\pi^*}(t) = \int_0^t e^{-\delta(s)} M_{\pi^*}(s) ds$$

and

$$P_{\pi^*}(t) = M_{\pi^*}(t) \left(\int_t^T e^{-\delta(s)} ds + K e^{-\delta(T)} \right),$$

we have

$$\mathbb{E}\left[\beta_0 (N_{\pi^*}(t+h) - N_{\pi^*}(t) + P_{\pi^*}(t+h) - P_{\pi^*}(t))\right] = 0.$$

Finally, using $Y_{\pi^*}(\cdot)$ defined as in equation (3.0.10), the equality becomes:

$$\mathbb{E}[\beta_0 (Y_{\pi^*}(t+h) - Y_{\pi^*}(t))] = 0.$$

Since this holds for all $\beta_0 \in \mathcal{G}_t$, we have :

$$\mathbb{E}[Y_{\pi^*}(t+h) | \mathcal{G}_t] = Y_{\pi^*}(t).$$

Hence $Y_{\pi^*}(t)$ is an $\mathcal{G}_t$-martingale for all $t \in [0,T]$. □

Theorem 3.0.16. *Suppose $\gamma(t,z) \neq 0$ and $\sigma(t) \neq 0$ for a.a. (t, ω). Suppose that there exist optimal consumption rate c_{λ^*} and optimal portfolio π^* for Problem 3.0.14. Then*

(i) *$B(t)$ is a $(\mathcal{G}_t, P)$-semimartingale. Therefore, there exists an adapted finite variation process $\alpha(t)$ such that*

$$\hat{B}(t) := B(t) - \int_0^t \alpha(s)ds$$

is a $\mathcal{G}_t$-Brownian motion.

(ii) *The process*

$$\int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \tilde{N}(d^- s, dz)$$

is a $\mathcal{G}_t$-semimartingale.

(iii) *The process*

$$\int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz)$$

is a $\mathcal{G}_t$-semimartingale.

(iv) *The optimal portfolio π satisfies the following equation :*

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \left(\int_s^T e^{-\delta(u)} du + K e^{-\delta(T)} \right) (\nu_{\mathcal{G}} - \nu_{\mathcal{F}})(ds,dz) \\
&+ \int_0^t \Bigg\{ \left(\mu(s) - r(s) - \sigma(s)^2 \pi(s) + \sigma(s)\alpha(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)} \nu_{\mathcal{F}}(dz) \right) \\
&\times \left(\int_s^T e^{-\delta(u)} du + K e^{-\delta(T)} \right) \Bigg\} ds = 0.
\end{aligned}$$

(v) *The optimal relative consumption rate is given by*

$$\lambda^*(t) = \frac{\mathbb{E}[e^{-\delta(t)} \mid \mathcal{G}_t]}{\mathbb{E}[\int_t^T e^{-\delta(s)} ds + K\, e^{-\delta(T)} \mid \mathcal{G}_t]}; \quad t \in [0,T].$$

Proof. Let (c_λ, π) be an optimal control of Problem 3.0.14. By the definition of $Y_\pi(t)$, we can write :

$$\begin{aligned}
Y_\pi(t) = &\int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)} \left(\int_s^T e^{-\delta(u)} du + K e^{-\delta(T)} \right) \tilde{N}(d^- s, dz) \\
&+ \int_0^t \sigma(s) \left(\int_s^T e^{-\delta(u)} du + K e^{-\delta(T)} \right) d^- B(s) \\
&+ \int_0^t \Bigg\{ \left(\mu(s) - r(s) - \sigma(s)^2 \pi(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)} \nu_{\mathcal{F}}(dz) \right) \\
&\times \left(\int_s^T e^{-\delta(u)} du + K e^{-\delta(T)} \right) \Bigg\} ds
\end{aligned}$$

and use the orthogonal decomposition into a continuous part $Y_\pi^c(t)$ and a discontinuous part $Y_\pi^d(t)$:

$$\begin{aligned} Y_\pi^c(t) &= \int_0^t \sigma(s)\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right) d^-B(s) \\ &\qquad - \int_0^t \sigma(s)\alpha(s)\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right) ds \\ Y_\pi^d(t) &= \int_0^t\int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)\tilde{N}(d^-s,dz) \\ &\qquad + \int_0^t\int_{\mathbb{R}_0}\theta(s,z)(\nu_{\mathcal{F}}-\nu_{\mathcal{G}})(ds,dz) \end{aligned}$$

where $\alpha(s)$ and $\theta(s,\cdot)$ are $\mathcal{G}$-adapted processes such that

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}_0}\theta(s,z)(\nu_{\mathcal{F}}-\nu_{\mathcal{G}})(ds,dz) - \int_0^t \sigma(s)\alpha(s)\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right) ds \\ &\quad = \int_0^t \Bigg\{\left(\mu(s)-r(s)-\sigma(s)^2\pi(s) - \int_{\mathbb{R}_0}\frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)}\nu_{\mathcal{F}}(dz)\right) \\ &\qquad \times\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)\Bigg\}ds. \end{aligned}$$

(i) For the continuous part $Y_\pi^c(t)$, we use the fact that

$$\int_0^t \frac{1}{\sigma(s)\left(\int_0^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)} dY_\pi^c(t) = B(t) - \int_0^t \alpha(s)ds$$

is a $\mathcal{G}_t$-martingale. Then we obtain directly that $B(t)$ is a $\mathcal{G}_t$-semimartingale.

(ii) Since $Y_\pi(t)$ is a $\mathcal{G}_t$-martingale, we can easily show that $\Gamma(t)$ defined as

$$\begin{aligned} \Gamma(t) :=& \int_0^t\int_{\mathbb{R}_0}\frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)\tilde{N}(d^-s,dz) \\ &+ \int_0^t \sigma(s)\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right) d^-B(s) \end{aligned}$$

is a $\mathcal{G}_t$-semimartingale. Then,

$$\begin{aligned} &\int_0^t \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)^{-1} d\Gamma(s) \\ &\qquad = \int_0^t\int_{\mathbb{R}_0}\frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}\tilde{N}(d^-,dz) + \int_0^t \sigma(s)d^-B(s) \end{aligned}$$

is also a $\mathcal{G}_t$-semimartingale. Finally, using (i)

$$\int_0^t\int_{\mathbb{R}_0}\frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}\tilde{N}(d^-s,dz)$$

is a $\mathcal{G}_t$-semimartingale.

(iii) By (ii), equation (1.0.4) and Remark 1.0.5, we know that

$$\int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}(\nu_{\mathcal{F}}-\nu_{\mathcal{G}})(ds,dz)$$

is of finite variation. Using Hypothesis (iv) of Definition 2.0.9, it follows that

$$\int_0^t \int_{\mathbb{R}_0} \gamma(s,z)(\nu_{\mathcal{F}}-\nu_{\mathcal{G}})(ds,dz)$$

is of finite variation. Since the $\mathcal{G}_t$-martingale $\int_0^t \int_{\mathbb{R}_0} \gamma(s,z)(N-\nu_{\mathcal{G}})(ds,dz)$ can be written as :

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}_0} \gamma(s,z)(N-\nu_{\mathcal{G}})(ds,dz)\\ &\qquad = \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)(\nu_{\mathcal{F}}-\nu_{\mathcal{G}})(ds,dz)\end{aligned}$$

and since $\int_0^t \int_{\mathbb{R}_0} \gamma(s,z)(\nu_{\mathcal{F}}-\nu_{\mathcal{G}})(ds,dz)$ is of finite variation then

$$\int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz)$$

is a $\mathcal{G}_t$-semimartingale.

(iv) We can write $Y_\pi(t)$ as :

$$\begin{aligned}Y_\pi(t) &= \int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)(N-\nu_{\mathcal{G}})(ds,dz)\\ &+ \int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)(\nu_{\mathcal{G}}-\nu_{\mathcal{F}})(ds,dz)\\ &+ \int_0^t \Bigg\{\left(\mu(s)-r(s)-\sigma(s)^2\pi(s)+\sigma(s)\alpha(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)}\nu_{\mathcal{F}}(dz)\right)\\ &\times\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)\Bigg\}ds + \int_0^t \sigma(s)\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)d\hat{B}(s).\end{aligned}$$

Hence by the martingale representation theorem, we have :

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}_0} \frac{\gamma(s,z)}{1+\pi(s)\gamma(s,z)}\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)(\nu_{\mathcal{G}}-\nu_{\mathcal{F}})(ds,dz)\\ &\quad + \int_0^t \left\{\mu(s)-r(s)-\sigma(s)^2\pi(s)+\sigma(s)\alpha(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\gamma(s,z)^2}{1+\pi(s)\gamma(s,z)}\nu_{\mathcal{F}}(dz)\right\}\\ &\quad \times\left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right)ds = 0.\end{aligned}$$

□

Finally, as a Corollary, let us present the results for an uninformed agent :

Corollary 3.0.17. *Suppose $\mathcal{F}_t = \mathcal{G}_t$, for all $t \in [0,T]$. Then the optimal portfolio $\pi(t)$ solves the following equation :*

$$\mu(t) - r(t) - \sigma(t)^2\pi(t) - \int_{\mathbb{R}_0} \frac{\pi(t)\gamma(t,z)^2}{1+\pi(t)\gamma(t,z)}\nu_{\mathcal{F}}(dz) = 0$$

and the optimal relative consumption rate $\lambda(t)$ is given by

$$\lambda(t) = \frac{\mathbb{E}[e^{-\delta(t)} \mid \mathcal{F}_t]}{\mathbb{E}[\int_t^T e^{-\delta(s)}ds + K\, e^{-\delta(T)} \mid \mathcal{F}_t]}; \quad t \in [0,T].$$

Proof. These results can be directly derived from Theorem 3.0.16. □

4. Examples

In this section, we give some examples to illustrate our results.

Example 1 : The Brownian motion case.
Suppose that $\gamma(t,z) = 0$ and $\sigma(t) \neq 0$. We denote by $\pi_i^*(t)$ and $\pi_h^*(t)$ the optimal portfolios for the insider and the uninformed agent respectively. By Theorem 3.0.16, the optimal portfolio $\pi_i^*(t)$ satisfies the following equation for all $t \in [0,T]$:

$$\int_0^t \left\{\mu(s) - r(s) - \sigma(s)^2\pi_i^*(s) + \sigma(s)\alpha(s)\right\} \left(\int_s^T e^{-\delta(u)}du + Ke^{-\delta(T)}\right) ds = 0.$$

Then we obtain an explicit solution for $\pi_i^*(t)$:

$$\pi_i^*(t) = \frac{\mu(t) - r(t)}{\sigma(t)^2} + \frac{\alpha(t)}{\sigma(t)}$$

and the optimal relative consumption rate for the insider $\lambda_i^*(t)$ is given by :

$$\lambda_i^*(t) = \frac{\mathbb{E}\left[e^{-\delta(t)}|\mathcal{G}_t\right]}{\mathbb{E}\left[\int_t^T e^{-\delta(s)}ds + Ke^{-\delta(T)}|\mathcal{G}_t\right]}.$$

For the uninformed agent, by Corollary 3.0.17, $\pi_h^*(t)$ and $\lambda_h^*(t)$ are given by :

$$\pi_h^*(t) = \frac{\mu(t) - r(t)}{\sigma(t)^2}, \qquad \lambda_h^*(t) = \frac{\mathbb{E}[e^{-\delta(t)} \mid \mathcal{F}_t]}{\mathbb{E}[\int_t^T e^{-\delta(s)}ds + K\, e^{-\delta(T)} \mid \mathcal{F}_t]}.$$

By (i) in Theorem 3.0.16, $B(t)$ is a $\mathcal{G}_t$-semimartingale then

$$\begin{aligned} X_i^{\left(c_{\lambda_i^*},\pi_i^*\right)}(t) =& X_h^{\left(c_{\lambda_h^*},\pi_h^*\right)}(t) \exp\left\{\int_0^t \left(\frac{1}{2}\alpha(s)^2 - \lambda_i^*(s) + \lambda_h^*(s)\right) ds \right. \\ &\left. + \int_0^t \alpha(s)d\hat{B}(s)\right\} \end{aligned} \tag{4.0.13}$$

where $X_i^{\left(c_{\lambda_i^*},\pi_i^*\right)}(t)$ and $X_h^{\left(c_{\lambda_h^*},\pi_h^*\right)}(t)$ are the optimal wealth processes for the insider and the uninformed agent respectively.

Hence,

$$
\begin{aligned}
J_i\left(c_{\lambda_i^*}, \pi_i^*\right) = J_h\left(c_{\lambda_h^*}, \pi_h^*\right) &+ \mathbb{E}\left[\int_0^T e^{-\delta(t)} \ln \frac{\lambda_i^*(t)}{\lambda_h^*(t)} dt\right] \\
&+ \mathbb{E}\left[\int_0^T e^{-\delta(t)} \int_0^t \left(\frac{1}{2}\alpha(s)^2 - \lambda_i^*(s) + \lambda_h^*(s)\right) ds\ dt\right] \\
&+ K\mathbb{E}\left[e^{-\delta(T)} \int_0^T \left(\frac{1}{2}\alpha(s)^2 - \lambda_i^*(s) + \lambda_h^*(s)\right) ds\right].
\end{aligned}
$$

Remark 4.0.18. If the discount rate $\delta(t)$ is deterministic, then

$$
\lambda_i^* = \frac{e^{-\delta(t)}}{\int_t^T e^{-\delta(s)} ds + K e^{-\delta(T)}} = \lambda_h^*.
$$

Note that although the optimal relative consumption rates are the same, the optimal consumptions are not the same among the informed and uninformed agent by equation (4.0.13).

If we restrict the enlarged filtration to be $\mathcal{G}_t = \mathcal{F}_t^B \vee \sigma(B(T_0)), T_0 > T$, then by equation (1.0.5)

$$
\alpha(t) = \frac{B(T_0) - B(t)}{T_0 - t}.
$$

and for $\delta(t) = 0$ the performance function of the informed agent can be written in terms of the performance function of the uninformed one as follows :

$$
\begin{aligned}
J_i\left(c_{\lambda_i^*}, \pi_i^*\right) &= J_h\left(c_{\lambda_h^*}, \pi_h^*\right) + \frac{1}{2}\int_0^T \int_0^t \frac{1}{T_0 - s} ds + \frac{K}{2}\int_0^T \frac{1}{T_0 - s} ds \\
&= J_h\left(c_{\lambda_h^*}, \pi_h^*\right) + \frac{1}{2}\Big[(T_0 - T)\ln(T_0 - T) + T\Big] + \frac{K}{2}\ln\left(\frac{T_0}{T_0 - T}\right).
\end{aligned}
$$

Example 2 : The mixed case.
Suppose that $\gamma(t, z) = z$ and $\sigma(t) \neq 0$. We consider the enlarged filtration $\mathcal{G}'_t = \mathcal{F}_t \vee \sigma(B(T_0), \eta(T_0)), T_0 > T$ and take the following assumptions :

(1) The informed agent has access to the filtration $\mathcal{G}_t$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{G}'_t$
(2) $\delta(\cdot)$ is a deterministic function.
(3) The Lévy measure $\nu_{\mathcal{F}}$ is given by $\nu_{\mathcal{F}}(ds, dz) = \rho\delta_1(dz)ds$ where $\delta_1(dz)$ is the unit point mass at 1.
(4) $\eta(t)$ is defined as $\eta(t) = Q(t) - \rho t$ with Q a Poisson process of intensity ρ.

Using the results of Di Nunno *et al.* (2006) Section 5, we obtain the following optimal portfolio $\pi_i^*(t)$:

$$
\pi_i^*(t) = \pi_h^*(t) + \frac{\zeta(t)}{\sigma(t)}
$$

with

$$
\begin{aligned}
\pi_h^*(t) &= \frac{\mu(t)-r(t)}{\sigma(t)^2} - \frac{\rho}{\sigma(t)^2},\\
\zeta(t) &= \frac{1}{2\sigma(t)}\Big[-\mu(t)+r(t)+\rho+\sigma(t)\alpha(t)-\sigma(t)^2\\
&\qquad + \sqrt{\left(\mu(t)-r(t)-\rho+\sigma(t)\alpha(t)+\sigma(t)^2\right)^2+4\sigma(t)^2\theta(t)}\Big],\\
\alpha(t) &= \frac{\mathbb{E}[B(T_0)-B(s)|\mathcal{G}_s]^-}{T_0-s},\\
\theta(t) &= \frac{\mathbb{E}[Q(T_0)-Q(s)|\mathcal{G}_s]^-}{T_0-s},
\end{aligned}
$$

where the notation $\mathbb{E}[...]^-$ denotes the left limit in s.
Moreover we have the optimal consumption rates $\lambda_i^*(t)$ and $\lambda_h^*(t)$ for the informed and uninformed agents respectively :

$$
\lambda_i^*(t) = \frac{e^{-\delta(t)}}{\int_t^T e^{-\delta(s)}ds + Ke^{-\delta(T)}} = \lambda_h^*(t).
$$

Subtituting these equalities into the wealth process equation and by Theorem 3.0.16 we can express the optimal wealth process of the informed agent in terms of the optimal wealth process of the uninformed agent :

$$
\begin{aligned}
X_i^{\left(c_{\lambda_i^*},\pi_i^*\right)}(t) =& X_h^{\left(c_{\lambda_h^*},\pi_h^*\right)}(t)\exp\Big\{\int_0^t\Big[-\frac{1}{2}\zeta(s)^2+\zeta(s)\alpha(s)\\
&+\int_{\mathbb{R}_0}\ln\left(1+\frac{z\zeta(s)\sigma(s)}{\sigma(s)^2+(\mu(s)-r(s)-\rho)z}\right)\nu_{\mathcal{G}}(dz)\Big]ds+\int_0^t\zeta(s)d\hat{B}(s)\\
&+\int_0^t\int_{\mathbb{R}_0}\ln\left(1+\frac{z\zeta(s)\sigma(s)}{\sigma(s)^2+(\mu(s)-r(s)-\rho)z}\right)(N-\nu_{\mathcal{G}})(ds,dz)\Big\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
J_i\left(c_{\lambda_i^*},\pi_i^*\right) =& J_h\left(c_{\lambda_h^*},\pi_h^*\right)+\mathbb{E}\left[\int_0^T e^{-\delta(t)}\int_0^t\left(-\frac{1}{2}\zeta(s)^2+\zeta(s)\alpha(s)\right)ds\,dt\right]\\
&+\mathbb{E}\left[\int_0^T e^{-\delta(t)}\int_0^t\int_{\mathbb{R}_0}\ln\left(1+\frac{z\zeta(s)\sigma(s)}{\sigma(s)^2+(\mu(s)-r(s)-\rho)z}\right)\nu_{\mathcal{G}}(dz)ds\,dt\right]\\
&+K\mathbb{E}\left[e^{-\delta(T)}\int_0^T\left(-\frac{1}{2}\zeta(s)^2+\zeta(s)\alpha(s)\right)ds\right]\\
&+K\mathbb{E}\left[e^{-\delta(T)}\int_0^T\int_{\mathbb{R}_0}\ln\left(1+\frac{z\zeta(s)\sigma(s)}{\sigma(s)^2+(\mu(s)-r(s)-\rho)z}\right)\nu_{\mathcal{G}}(dz)ds\right].
\end{aligned}
$$

By Proposition 5.2 in [7], $\nu_{\mathcal{G}}(dz)ds$ can be also expressed as

$$
\nu_{\mathcal{G}}(dz)ds = \mathbb{E}\left[\frac{1}{T_0-s}\int_s^{T_0}N(dr,dz)|\mathcal{G}_s\right]ds.
$$

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